25

JPRS: 4659

29 May 1961

THE MAXIMUM PRINCIPLE IN THE THEORY OF OPTIMAL PROCESSES

By V. G. Boltyanskiy

-USSR-

Distributed by:

OFFICE OF TECHNICAL SERVICES
U. S. DEPARTMENT OF COMMERCE
WASHINGTON 25, D. C.

U. S. JOINT PUBLICATIONS RESEARCH SERVICE
1636 CONNECTICUT AVE., N.W.
WASHINGTON 25, D. C.

F O R E W O R D

This publication was prepared under contract by the UNITED STATES JOINT PUBLICATIONS RESEARCH SERVICE, a federal government organization established to service the translation and research needs of the various government departments.

JPRS: 4659

CSO: 1797-S

THE MAXIMUM PRINCIPLE IN THE THEORY OF OPTIMAL PROCESSES

-USSR-

[Following is the translation of an article by V. G. Boltyanskiy in Doklady Akademii Nauk SSSR (Reports of the Academy of Sciences USSR), Vol 119, No 6, Moscow, 1958, pages 1070-1073.]

(Presented by Academician P. S. Aleksandrov, 19 December 1957.)

This paper borders on our joint work [1] on the theory of optimal processes. The investigations of the secondary variations by R. V. Gamkrelidze led to Pontryagin's hypothesis of the maximum principle. Here, complete proof of the maximum principle as a necessary condition of optimum is presented. To obtain this result the whole work is repeated with different excitations (variations).

Here is considered a point of motion $x = (x^1,\dots,x^n)$ in a n-dimensional phase space X described by a law:

$$\dot{x} = f^i(x^1,\dots,x^n, u) = f^i(x, u), \quad i = 1,\dots,n, \tag{1}$$

where u is a control parameter which varies within the boundries of a certain domain U. The domain U may be any topological space, for instance, any arbitrary bounded set of the r-dimensional space of the variable $u = (u^1,\dots,u^r)$; the functions f^i are assumed to be continuously dependent on two variables x, u and continuously differentiable with respect to $x^1,\dots,x^n$. If the law of the control is known, i.e., if it is given the variable $u(t) \in U$, then the system (1) uniquely describes the law of motion of this variable point; the control $u(t)$ must be piece-wise continuous. Similarly with earlier work [1] the problem is stated:

"Find a control $u(t)$ which secures the minimal time for the process of transfer of a point x from a given initial coordinate $x_0 \in X$". The control defined above and the trajectory corresponding

to it are called optimal control and optimal trajectory. Let $u(t)$, $t_0 \leqslant t \leqslant t_1$, be a certain control within the domain U. We will choose the instants of time $\tau_1, \ldots, \tau_k$, where $t_0 < \tau_1 < \ldots < \tau_k < t_1$ and let $\delta t_{i,j}$, $j = 1, \ldots, s_i$, $i = 1, \ldots, k$, be arbitrary non-negative numbers and $v_{i,j}$ be arbitrary points of the domain U. We will denote by $I_{i,j}$ a segment

$$\tau_i + \varepsilon(\delta t_{i,1} + \ldots + \delta t_{i,j-1}) \leqslant t < \tau_i + \varepsilon(\delta t_{i,1} + \ldots + \delta t_{i,j})$$

and assume

$$\bar{u}(t) = \begin{cases} v_{i,j} & \text{if } t \in I_{i,j}; \\ u(t), & \text{if } t \text{ does not belong to any segment } I_{i,j}. \end{cases}$$

Here ε is a positive quantity which affects the control $\bar{u}(t)$. Subsequently we will consider ε as a variable of the first order of infintesimality; the quantities of higher order of infinitesimality will be deonoted by a row of dots and ignored. We will note that the control $\bar{u}(t)$ is derived by variations of control $u(t)$ in the neighborhood of the instants of time $\tau_1, \ldots, \tau_k$. We will denote by $\bar{x}(t)$ a trajectory originated at the same coordinate x_0 and corresponding to varied control $\bar{u}(t)$ and we will consider a vector

$$\bar{x}(t_1 - \varepsilon\,\delta t) - x(t_1), \qquad (2)$$

where δt is a non-negative number.
The linear with respect to ε part of the vector (2) we will denote by $\varepsilon\,\Delta x$:

$$\bar{x}(t_1 - \varepsilon\,\delta t) = x(t_1) + \varepsilon\,\Delta x + \ldots \qquad (3)$$

If we consider now all possible control $\bar{u}(t)$ resulting from variation of the control $u(t)$ and independently from this we vary δt giving it all possible non-negative values, then the vectors Δx, originated in the point $x_1 = x(t_1)$, form a certain cone with its vertex in the point x_1. By means of fairly simple logic the following statement could be proven: Lemma. "The cone of attainability is a convex cone". Subsequently the theorem presented is an intensification of the theorem in [1], which states that dimension of a linear manifold P' constructed there is not greater than $n - 1$.

Theorem 1. "If a cone of attainability K fulfills the whole space X of the variables $x^1, \ldots, x^n$, then the trajectory $x(t)$ and the corresponding control $u(t)$, $t_0 \leqslant t \leqslant t_1$ are not optimal."

Thus, if $x(t)$ and $u(t)$ are optimal than a convex cone K cannot occupy the whole space X and therefore the whole cone K lies in one half-space describable by a certain hyperplane passing through a point x_1. Let $a_\alpha(x^\alpha - x_1^\alpha) = 0$ be the equation of this hyperplane. We assume that the whole cone K lies in a negative half-space $a_\alpha(x^\alpha - x_1^\alpha) \leqslant 0$; we may change the algebraic signs of all coefficients a_α, if necessary.

If we assume, just as in [1],

$$a_\alpha \varphi_\beta^\alpha(t_1) = b_\beta, \quad b_\beta \psi_\gamma^\beta(t) = \psi_\gamma(t),$$

then we receive $\psi_\alpha(t_1) = a_\alpha$ and the equation of a negative half-space will have a mode $\psi_\alpha(t_1)\ (x^\alpha - x_1^\alpha) \leqslant 0.$

Consequently we will get the following necessary condition for optimum:

$$\psi_\alpha(t_1)\,\Delta x^\alpha \leqslant 0 \tag{4}$$

where Δx is determined by equation (3) for any varied control $\bar{u}$ (t) and any $\delta t \geqslant 0$.

This condition we may regard as a generalization of the conditions (6) and (7) of [1]. In the work presented in [1] the variations of $\delta u(t)$ also produced a certain cone of attainability, which however may be smaller than the cone K discussed here. Consequently it is possible to have a case when quantities a_α and functions $\psi_\alpha(t)$ would satisfy all the condistions imposed in the [1] and would not conform with the condition (4). In other words, the condition (4) is more strict than the conditions (6) and (7) of [1].

From (4) it is easy to deduce the following inequality

$$\psi_\alpha(t_1)\, f^\alpha(x(t_1), u(t_1)) \geqslant 0,$$

or, we may say the function $H(x, \psi, u) = \psi_\alpha f^\alpha(x,u)$ will have a non-negative value at the point $(x\ (t_1),\ \psi(t_1),\ u\ (t_1))$. L. S. Pontryagin pointed out that the function $H(x, \psi, u)$ plays an important role in the theory of optimal processes. In particular, the relations (1) and (5) of [1] may be written in the form

$$\dot{x}^i = \frac{\partial H}{\partial \psi_i}, \quad \dot{\psi}_i = -\frac{\partial H}{\partial x^i}; \quad i = 1, \ldots, n. \tag{5}$$

Theorem 2. (L. S. Pontryagin's Maximum Principle). "For optimal trajectory x(t) and control u(t), $t_0 \leqslant t \leqslant t_1$, it is necessary to have such variable vector $\psi(t)$, that the relations (5) will be satisfied and besides with any t, $t_0 \leqslant t \leqslant t_1$, the following condition will be also satisfied

$$H(x\ (t), \psi(t), u(t)) = \max_{u \in U} H(x(t), \psi(t), u) \geqslant 0.$$

A sketchy proof of this theorem follows. We will choose a vector $\psi(t)$ as outlined above. We assume that for a certain τ, $t_0 < \tau < t_1$, and a certain $v \in U$ we will have

$$H(x(\tau), \psi(\tau), u(\tau)) < H(x(\tau), \psi(\tau), v).$$

We will vary the control u(t) on the single interval $I_{1,1}$ of the duration $\varepsilon\delta t_{1,1}$ which starts at τ, and we will assume $v_{1,1} = v$. Then we will have: $\bar{x}(t) = x(t)$ with $t_0 \leqslant t \leqslant \tau$ and

$$\bar{x}(t) - x(t) = [f(x(\tau), v) - f(x(\tau), u(\tau))](t - \tau) + \ldots$$

with $\tau \leqslant t \leqslant \tau + \varepsilon\delta t_{1,1}$.

Particularly,

$$\bar{x}(\tau+\varepsilon\delta t_{1,1})-x(\tau+\varepsilon\delta t_{1,1})=[f(x(\tau),v)-f(x(\tau),u(\tau))]\varepsilon\delta t_{1,1}+\ldots,$$

from which it follows, in accordance with definition of H, that

$$\psi_\alpha(\tau+\varepsilon\delta t_{1,1})[\bar{x}^\alpha(\tau+\varepsilon\delta t_{1,1})-x^\alpha(\tau+\varepsilon\delta t_{1,1})]=A\varepsilon+\ldots, \qquad (6)$$

where $A>0$,

Since on the interval $\tau+\varepsilon\delta t_{1,1}<t\leqslant t_1$ the control $\bar{u}(t)$ coincides with $u(t)$, then on this interval we will have:

$$\bar{x}^\alpha(t)=x^\alpha(t)+\varepsilon\delta x^\alpha(t)+\ldots, \quad \text{where}$$

$$\delta\dot{x}^i(t)=\frac{\partial f^i(x(t),u(t))}{\partial x^\alpha}\delta x^\alpha(t), \quad i=1,\ldots,n.$$

Thus, with $\tau+\varepsilon\delta t_{1,1}\leqslant t\leqslant t_1$ in accordance with the second part of (5) we will have:

$$\frac{d}{dt}(\psi_\alpha(t)\delta x^\alpha(t))=\dot{\psi}_\alpha(t)\delta x^\alpha(t)+\psi_\alpha(t)\delta\dot{x}^\alpha(t)=$$

$$=-\frac{\partial f^\beta}{\partial x^\alpha}\psi_\beta(t)\delta x^\alpha(t)+\psi_\alpha(t)\frac{\partial f^\alpha}{\partial x^\beta}\delta x^\beta(t)=0.$$

Therefore on the whole interval $\tau+\varepsilon\delta t_{1,1}\leqslant t\leqslant t_1$ it holds the relation

$$\psi_\alpha(t)\delta x^\alpha(t)=\psi_\alpha(\tau+\varepsilon\delta t_{1,1})\delta x^\alpha(\tau+\varepsilon\delta t_{1,1})=A>0$$

see (6), and consequently

$$\psi^\alpha(t_1)[\bar{x}^\alpha(t_1)-x^\alpha(t_1)]=\psi_\alpha(t_1)[\varepsilon\delta x^\alpha(t_1)+\ldots]=A\varepsilon+\ldots$$

Hence, the vector $\Delta x=\delta x(t_1)$, derived according to formulas (2) and (3) with $\delta t=0$, what is permissible, satisfies the condition $\psi_\alpha(t_1)\Delta x^\alpha=A>0$; but this contradicts the condition (4).

Closely related to the Maximum Principle is the following important property of the function H, which states that along the optimal trajectory $H\geqslant0$, i.e. the maximum, the existance of which was proven, is non-negative.

Theorem 3. If $x(t)$ is an optimal trajectory, $u(t)$ $t_0\leqslant t\leqslant t_1$ is an optimal control, and $\psi(t)$ is a variable vector, the existence of which is confirmed in theorem 2, then the function H maintains along $x(t)$, $\psi(t)$, $u(t)$ a constant non-negative value:

$$H(x(t),\psi(t),u(t))=\text{const.}$$

This relation is proven first for every segment of piece-wise continuous control $u(t)$. Subsequently it is proven that at every point of discontinuity of the control $u(t)$ the function H is not changing its value.

These results are obtained in a seminar on the theory of oscillations and automatic control personally conducted by L. S. Pontryagin. L. S. Pontryagin pointed out to me one possible simplification in my derivation of the Maximum Principle proof. Owing to that suggestion my proof, as it is now, holds for any arbitrary topological space U; when my original variant had a redundant construction not used for any useful purpose yet limiting the proof to the case of U being bounded domain of the vector space with piece-wise-smooth frontier and convex interior angles at the joints.

V. A. Steklov's Institute of
Mathematics of the Academy
of Sciences USSR.

Received
18 December 1957

REFERENCES

1. V. G. Boltyansky, R. V. Gamkrelidze and L. S. Pontryagin, Doklady Akademii Nauk SSSR (Reports of the Academy of Sciences USSR), Vol 110, No 1, 1956, p 7-11.